I) A_∞-categories

Def: $\mathcal{A}$ A_∞-category (not necess. unital) over a coeff. field $\mathbb{K}$:=

- set of objects $Ob\,\mathcal{A}$
- graded vector spaces $hom_{\mathcal{A}}(X,Y)$ $\forall X,Y \in Ob\,\mathcal{A}$
- $\forall d \geq 1$, $\mu^d_{\mathcal{A}} : hom_{\mathcal{A}}(X_{d-1},X_d) \otimes \dots \otimes hom_{\mathcal{A}}(X_0,X_1) \to hom_{\mathcal{A}}(X_0,X_d)[2-d]$

shift grading by $2-d$: $\deg \mu^d(a_d .. a_1) = \sum \deg a_i + 2 - d$.

s.t. $\forall a_1 \dots a_d$, $\displaystyle\sum_{\substack{1\leq l\leq d\\ 0\leq k\leq d-l}} (-1)^{*} \mu_{\mathcal{A}}^{d-l+1}(a_d,\dots,a_{k+l+1},\mu^l_{\mathcal{A}}(a_{k+l},\dots,a_{k+1}),a_k \dots a_1) = 0$.

w/ sign. $* = \deg(a_1) + \dots + \deg(a_k) - k$.

So:
- $\mu^1(\mu^1(a)) = 0$: μ^1 is a differential on hom spaces
- $\mu^1(\mu^2(b,a)) = \pm\mu^2(\mu^1(b),a) \pm \mu^2(b,\mu^1(a))$ Leibniz rule
- $\mu^2(c,\mu^2(b,a)) \pm \mu^2(\mu^2(c,b),a) = \pm\mu^1(\mu^3(c,b,a)) \pm \mu^3(\mu^1 c, b, a) \pm \mu^3(c,\mu^1 b,a) \pm \mu^3(c,b,\mu^1 a)$

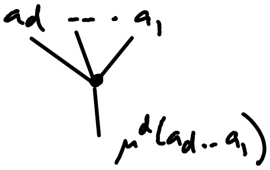
μ^2 is associative up to a homotopy given by μ^3.

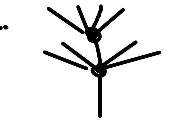
...

Rmk:
- key point: in real life, at chain level things aren't associative on the nose, only up to homotopy.
- taking cohomology wrt μ^1, get an honest graded category $\underline{\underline{H(\mathcal{A})}}$ on which composition is associative. $hom_{H(\mathcal{A})}(X,Y) = H^*(hom_{\mathcal{A}}(X,Y),\mu^1)$.

  except signs are funny $\Rightarrow$ set $[a_2].[a_1] = (-1)^{\deg a_1}[\mu^2(a_2,a_1)]$.

  (also, $H^0(\mathcal{A})$ = only degree 0 cohom. = ordinary category).
- A_∞-cat. with $\mu^d \equiv 0$ $\forall d \geq 3 \iff$ dg-category (up to changing signs...)
- μ^d $d\geq 3$ partially survive to $H(\mathcal{A})$ as Massey products.

• Pictorial representation: 

; A_∞-eq^{ns}. $\sum (-1)^{\cdots}$  $=0$ ②

• Bar complex: $T(\mathcal{A}[1]) = \bigoplus_{d\geq 0} \mathcal{A}[1]^{\otimes d} := \bigoplus_d \bigoplus_{\substack{X_1,\dots X_d \\ \in \mathrm{Ob}\,\mathcal{A}}} \hom(X_{d-1},X_d)\otimes\dots\otimes\hom(X_0,X_1)[d]$

$\circlearrowleft$

$$\delta(a_d\otimes\dots\otimes a_1) := \sum_{k,l} (-1)^{*} a_d\otimes\dots\otimes a_{k+l+1}\otimes\mu^l(a_{k+l},\dots,a_{k+1})\otimes\dots\otimes a_1$$

Then $\{\mu^d\}$ A_∞-relations $\iff$ δ is a deg. 1 differential on $T(\mathcal{A}[1])$.
$(\delta^2=0)$

• Examples: • Morse A_∞-cat. of a Riem. mfld (Fukaya)
• Fukaya cat. of a sympl. manifold.

II) Cohomological unitality:

• say $\mathcal{A}$ strictly unital if $\forall X\in \mathrm{Ob}\,\mathcal{A}$, $\exists\, e_X\in\hom^0(X,X)$ st.
- $\mu^1(e_X)=0$
- $\forall a\in\hom(X,Y)$, $(-1)^{\deg a}\mu^2(e_Y,a) = a = \mu^2(a,e_X)$
- $\mu^d(\dots,e_X,\dots)\equiv 0$ whenever $d\geq 3$.

Too strong to be useful in real-world.

• weaker: $\mathcal{A}$ has homotopy units, ie. $e_X\in\hom^0(X,X)$ satisfies $\mu^1(e_X)=0$; $\mu^2(a,e_X) = a \pm \mu^1(h_X^r(a)) \pm h_X^r(\mu^1(a))$
h_X^r homotopy for e_X right unit
similarly $\mu^2(e_Y,a)$ homotopy h_Y^l
... higher homotopies ... quite painful.

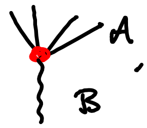
• even weaker but more tractable & sufficient: cohomology units ie.
$e_X\in\hom^0(X,X)$, $\mu^1(e_X)=0$, $[e_X]$ is a unit in $H(\mathcal{A})$.

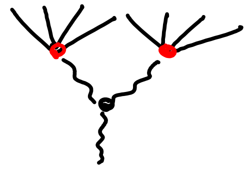
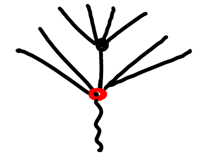
III) A_∞-functors: $\mathcal{F}:\mathcal{A}\to\mathcal{B}$ consists of a map $\mathcal{F}:\mathrm{Ob}\,\mathcal{A}\to\mathrm{Ob}\,\mathcal{B}$

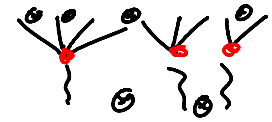
and multilinear $\mathcal{F}^d : \hom_{\mathcal{A}}(X_{d-1},X_d)\otimes\dots\otimes\hom_{\mathcal{A}}(X_0,X_1)\to\hom_{\mathcal{B}}(\mathcal{F}(X_0),\mathcal{F}(X_d))[1-d]$

st. $\sum_r \sum_{s_1+\dots+s_r=d} \mu_{\mathcal B}^r\left(\mathcal F^{s_r}(a_d \dots a_{d-s_r+1}), \dots, \mathcal F^{s_1}(a_{s_1},\dots a_1)\right)$

$$= \sum_{k,l} (-1)^* \mathcal F^{d-l+1}\left(a_d,\dots,a_{k+l+1}, \mu_{\mathcal A}^l(a_{k+l}\dots a_{k+1}), a_k \dots a_1\right)$$

Pictorially: $\mathcal F^d =$ 

, then says $\sum$  $\pm$  $= 0$

★ Equivalently: on bar complex, $T(\mathcal F): T(\mathcal A[1]) \longrightarrow T(\mathcal B[1])$ 
(Σ: group inputs arbitrarily & apply $\mathcal F$ to each group)

$\mathcal F$ <u>A_∞-functor</u> $\Leftrightarrow$ $T(\mathcal F)$ <u>chain map</u>.

★ Observe: $\mu_{\mathcal B}^2(\mathcal F^1(b), \mathcal F^1(a)) + \mu_{\mathcal B}^1(\mathcal F^2(b,a)) = \mathcal F^1(\mu_{\mathcal A}^2(b,a)) \pm \mathcal F^2(b,\mu^1 a)$
$\pm \mathcal F^2(\mu^1 b, a)$

$\Rightarrow \mathcal F^1$ usual functor up to homotopy given by $\mathcal F^2$ etc ...

in particular:
$$H(\mathcal F): H(\mathcal A) \to H(\mathcal B)$$
$$X \longmapsto \mathcal F(X)$$
$$[a] \longmapsto [\mathcal F^1(a)]$$
is an <u>ordinary functor</u>.

★ say $\mathcal F$ <u>strictly unital</u> if $\mathcal F^1(e_X) = e_{\mathcal F(X)}$; <u>c-hom-unital</u> if $H(\mathcal F)$ is unital
$\mathcal F^d(\dots, e_X \dots) = 0 \ \forall d \geq 2$

★ <u>Def</u>: $\mathcal F$ is a <u>quasi-isomorphism</u> if $H(\mathcal F)$ is an isomorphism (ie. $\exists G$, $FG = \mathrm{id}$, $GF = \mathrm{id}$)
$\mathcal F$ <u>c-unital</u> is a <u>quasi-equiv^ce</u> if $H(\mathcal F)$ equivalence (of unital cats.) $\to$ ($\Leftrightarrow$ full, faithful, ess. surjective)

★ <u>Composition of A_∞-functors</u>: $(\mathcal G \circ \mathcal F)^d(a_d \dots a_1) = \sum_{\substack{r \\ s_1+\dots+s_r=d}} \mathcal G^r\left(\mathcal F^{s_r}(a_d \dots), \dots, \mathcal F^{s_1}(\dots, a_1)\right)$
is strictly associative !!

★ given a functor $F: H(\mathcal A) \to H(\mathcal B)$, can understand obstruction to $\exists$ A_∞-functor $\mathcal F$ st. $H(\mathcal F) = F$ - lives in a certain Hochschild cohomology group.

★ A_∞-functors $\mathcal A \to \mathcal B$ are the objects of an A_∞-cat. $\mathrm{Fun}(\mathcal A, \mathcal B)$ with morphisms = A_∞-pre-<u>natural transformations</u>: $T \in \hom(\mathcal F, \mathcal G)$ = not just collection of maps $T^0 \in \hom_{\mathcal B}(\mathcal F(X), \mathcal G(X))$ $\forall X \in \mathrm{ob}\,\mathcal A$
but also $T^d: \hom_{\mathcal A}(X_{d-1}, X_d) \otimes \dots \otimes \hom_{\mathcal A}(X_0, X_1) \to \hom_{\mathcal B}(\mathcal F(X_0), \mathcal G(X_d))$

... <u>A_∞-nat. transf.</u> if well-behaved ($\Leftrightarrow$ closed under $\mu^1_{\mathrm{Fun}(\mathcal A, \mathcal B)}$)

* A_∞-functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$ which coincide on objects are homotopic if the natural transform^n $\mathcal{D} = \mathcal{F} - \mathcal{G}$: $\left(\mathcal{D}^0 = 0,\ \mathcal{D}^d = \mathcal{F}^d - \mathcal{G}^d \ \forall d \geq 1\right)$ is nullhomotopic $\mathcal{D} = \mu^1_{Fun(\mathcal{A},\mathcal{B})}(T)$.

IV. Homological perturbation lemma:

$\mathcal{B}$ A_∞-cat., assume $\forall X, Y \in Ob\,\mathcal{B}$ we have
- a chain complex $(A(X,Y), \delta_A)$
- chain maps of degree 0 $F \downarrow \uparrow G$
- a homotopy $T \circlearrowleft_{[-1]} (\hom_{\mathcal{B}}(X,Y), \mu^1_{\mathcal{B}})$ between $F \circ G$ and Id, i.e.

$$FG - Id = \mu^1_{\mathcal{B}} \circ T + T \circ \mu^1_{\mathcal{B}}$$

Then $\exists$ A_∞-cat. $\mathcal{A}$ with $Ob\,\mathcal{A} = Ob\,\mathcal{B}$

$$\hom_{\mathcal{A}}(X,Y) = A(X,Y), \quad \mu^1_{\mathcal{A}} = \delta_A$$

& A_∞-functors $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ which are id on objects
& s.t. $\mathcal{F}^1 = F$, $\mathcal{G}^1 = G$
& s.t. $\mathcal{F} \circ \mathcal{G}$ is homotopic to Id.

(& can determine everything, esp $\mu^d_{\mathcal{A}}$, by explicit recursive formulas).

Corollary: $\mathcal{B}$ A_∞-cat. $\Rightarrow$ $\exists$ quasi-isomorphic A_∞-cat $\mathcal{A}$ with $\mu^1_{\mathcal{A}} = 0$ ("minimal model" of $\mathcal{B}$)

$[\,Ob\,\mathcal{A} = Ob\,\mathcal{B},\ \hom_{\mathcal{A}}(X,Y) \cong H^*(\hom_{\mathcal{B}}(X,Y), \mu^1_{\mathcal{B}}),\ \mu^1_{\mathcal{A}} = 0,\ \mu^2_{\mathcal{A}} = \mu^2_{H(\mathcal{B})}\,]$

but $\mu^{\geq 3}_{\mathcal{A}}$ may be nonzero even if $\mathcal{B}$ is dg-cat.

Corollary: Any quasi isom. of A_∞-categories has an inverse up to homotopy.
[by passing to minimal models where it becomes an iso...].

Also, **Lemma:** $\mathcal{A}$ cohomologically unital $\Rightarrow$ $\exists$ $\tilde{\mathcal{A}}$ strictly unital A_∞-cat. with same objects & morphisms as $\mathcal{A}$ & quasi isom. $\phi : \mathcal{A} \xrightarrow{\sim} \tilde{\mathcal{A}}$ $\begin{cases} \phi = id \text{ on Ob} \\ \phi^1 = id \text{ on Hom} \end{cases}$

V) A_∞-modules (right modules) (not necess. unital) / A_∞-cat. $\mathcal{A}$

= contravariant functor $\mathcal{A} \to$ chain complexes; more explicitly:

$\mathcal{M} \in \text{mod-}\mathcal{A} \iff$
- $\forall X \in \text{Ob}\,\mathcal{A}$, $\mathcal{M}(X)$ graded vector space
- $\forall d \geq 1$, $\mu^d_{\mathcal{M}} : \mathcal{M}(X_d) \otimes \hom(X_{d-1}, X_d) \otimes \dots \otimes \hom(X_1, X_2) \to \mathcal{M}(X_1)[2-d]$

eq^{n}:
$$\sum_k (-1)^* \mu^{d-k+1}_{\mathcal{M}}\left(\mu^k_{\mathcal{M}}(m, a_d \dots a_{d-k+1}), a_{d-k} \dots a_1\right) + \sum_{k,\ell} (-1)^* \mu^{d-\ell+1}_{\mathcal{M}}\left(m, a_d \dots \mu^\ell_{\mathcal{A}}(a_{k+\ell} \dots a_{k+1}), a_k \dots a_1\right) = 0$$

In particular $\mu^1_{\mathcal{M}}(\mu^1_{\mathcal{M}}(m)) = 0$ differential

$$\mu^1_{\mathcal{M}}(\mu^2_{\mathcal{M}}(m,a)) = \pm \mu^2_{\mathcal{M}}(\mu^1_{\mathcal{M}}(m), a) \pm \mu^2_{\mathcal{M}}(m, \mu^1_{\mathcal{A}}(a))$$

Leibniz rule

$$\mu^2_{\mathcal{M}}(m, \mu^2_{\mathcal{A}}(b,a)) \pm \mu^2_{\mathcal{M}}(\mu^2_{\mathcal{M}}(m,b), a) = \pm \mu^1_{\mathcal{M}}(\mu^3_{\mathcal{M}}(m,b,a)) + \dots$$

product $\mu^2_{\mathcal{M}}$ is assoc. up to homotopy $\mu^3_{\mathcal{M}}$ ...

(if $\mu^{\geq 3} = 0$, $\leftrightarrow$ dg-module over dg-cat.)

- In terms of bar complex:

$$\mathcal{M} \otimes T(\mathcal{A}[1]) = \bigoplus_d \bigoplus_{X_1 \dots X_d \in \text{Ob}\,\mathcal{A}} \mathcal{M}(X_d) \otimes \hom(X_{d-1}, X_d) \otimes \dots \otimes \hom(X_1, X_2)[d]$$

$$\delta_{\mathcal{M}} = \sum (-1)^* \mu_{\mathcal{M}} \otimes id + \sum (-1)^* id \otimes \mu_{\mathcal{A}} \otimes id$$

eq^{ns} say $\delta^2_{\mathcal{M}} = 0$.

- Pre-module homomorphisms:

$\hom_{\text{mod-}\mathcal{A}}(\mathcal{M}, \mathcal{N}) \ni t = \{t^d\}_{d \geq 1}$, $t^d : \mathcal{M}(X_d) \otimes \hom(X_{d-1}, X_d) \otimes \dots \otimes \hom(X_1, X_2) \to \mathcal{N}(X_1)[\deg t + 1 - d]$

(without any condition) $\iff$ induces $\mathcal{M} \otimes T(\mathcal{A}[1]) \to \mathcal{N} \otimes T(\mathcal{A}[1])$

- diff^{l}: $\mu^1_{\text{mod}}(t)$ vanishes $\iff$ t is a module homomorphism.

$\iff t$ induces chain map $\mathcal{M} \otimes T(\mathcal{A}[1]) \to \mathcal{N} \otimes T(\mathcal{A}[1])$.

$$\left(\mu^1_{mod}(t)\right)^d(m, a_{d-1}\dots a_1) = \sum (-1)^{\dots} \mu^{k+1}_{\mathcal{N}}\left(t^{d-k}(m, a_{d-1}\dots a_{k+1}), a_k \dots a_1\right)$$
$$+ \sum (-1)^{\dots} t^{k+1}\left(\mu^{d-k}_{\mathcal{M}}(m, a_{d-1}\dots a_{k+1}), a_k \dots a_1\right)$$
$$+ \sum (-1)^{\dots} t^{k+1}\left(m, \dots, \mu^{d-k}_{\mathcal{A}}(\dots), \dots a_1\right).$$

- composition: $\Leftrightarrow$ composition of maps on bar complexes

hence $\mu^2_{mod}(t_2, t_1)^d(m, a_{d-1}\dots a_d) = \sum (-1)^{\dots} t_2^{k+1}\left(t_1^{d-k}(m, a_{d-1}\dots a_{k+1}), a_k \dots a_1\right)$

$\mathcal{M} \xrightarrow{t_1} \mathcal{N} \xrightarrow{t_2} \mathcal{P}$

Observe: μ^2_{mod} is strictly associative! set $\mu^{\geq 3}_{mod} \equiv 0$. (i.e. A_∞-mod/$\mathcal{A}$ form a dg-category!)

Observe: mod-$\mathcal{A}$ is strictly unital ($id_{\mathcal{M}}$ has $id^1 = id$, $id^d = 0 \; \forall d \geq 2$)

VI) Yoneda embedding: = canonical A_∞-functor $\mathcal{A} \xrightarrow{\mathcal{I}}$ mod-$\mathcal{A}$.

Def: on objects: $Y \in \text{ob}\,\mathcal{A} \mapsto \mathcal{Y} = \mathcal{I}(Y) \in$ mod-$\mathcal{A}$ defined by

$\mathcal{Y}(X) = \hom_{\mathcal{A}}(X, Y)$, and structure maps = right multiplications in $\mathcal{A}$:

$$\mu^d_{\mathcal{Y}}(\underset{\hom_{\mathcal{A}}(X_d, Y)}{m}, \underset{\hom_{\mathcal{A}}(X_{d-1}, X_d)}{a_{d-1}} \dots \underset{\hom_{\mathcal{A}}(X_1, X_2)}{a_1}) := \mu^d_{\mathcal{A}}(m, a_{d-1}\dots a_1).$$

on morphisms: $b \in \hom_{\mathcal{A}}(Y_0, Y_1) \rightsquigarrow \mathcal{I}^1(b) \in \hom_{mod}(\mathcal{Y}_0, \mathcal{Y}_1)$

$$\mathcal{I}^1(b)^{(d)}(\underset{\hom_{\mathcal{A}}(X_d, Y_0)}{m}, a_{d-1}\dots a_1) = \underset{\in \hom_{\mathcal{A}}(X_1, Y_1)}{\mu^{d+1}_{\mathcal{A}}(b, m, a_{d-1}\dots a_1)}$$

& higher terms similarly $\mathcal{I}^k(b_k \dots b_1)^{(d)}(m, a_{d-1}\dots a_1) = \mu^{d+k}_{\mathcal{A}}(b_k \dots b_1, m, a_{d-1}\dots a_1)$

- This has pretty much every naturality property one can wish for
- Thm: $\mathcal{I}$ is full and faithful on cohomology & c-unital (hence "embedding") (i.e. $\forall X, Y$ $H^*\hom_{\mathcal{A}}(X,Y) \xrightarrow[\simeq]{[\mathcal{I}^1]} H^*\hom_{mod}(\mathcal{X}, \mathcal{Y})$)
- Corollary: any c-unital A_∞-cat. is quasi-isom. to a strictly unital dg-category. (namely, image of $\mathcal{I}$).
- mod-$\mathcal{A}$ usually a lot larger than $\mathcal{A}$! can take $\oplus$ of objects, mapping cones, ...!